Section 16.2: Line Integrals

What We'll Learn In Section 16.2

1. Line Integrals (with respect to arclength)
2. Line Integrals (with respect to x and y)
3. Line Integrals in Space
4. Line Integrals of Vector Fields

1. Line Integrals (with respect to arclength)

Situation:

- Let C be a curve in the xy-plane with parametrization

$$x = x(t)\ \ ,\ \ y = y(t)\ \ ,\ \ a \le t \le b$$

- Let $f(x, y)$ be a 2-variable function defined on the curve C

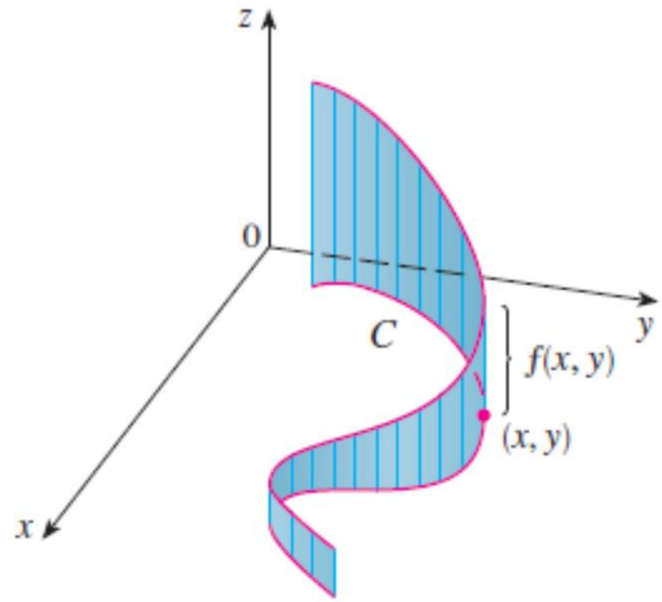
1. Line Integrals (with respect to arclength)

2 Definition

If f is defined on a smooth curve C, then the **line integral of f along C** is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*, y_i^*\right)\Delta s_i$$

if this limit exists.



3 $$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

1. Line Integrals (with respect to arclength)

Ex 1: Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle

$x^2 + y^2 = 1$.

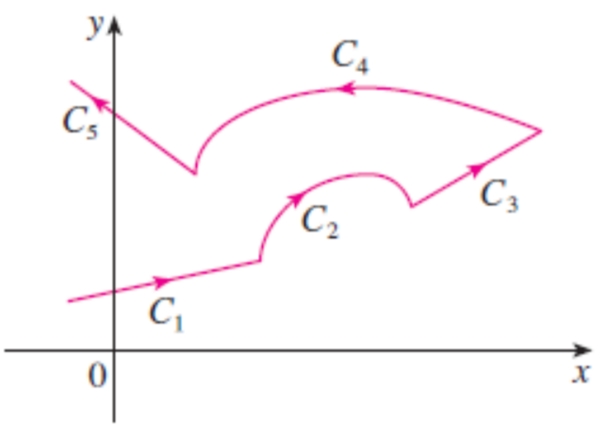
1. Line Integrals (with respect to arclength)

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

Figure 4

A piecewise-smooth curve



1. Line Integrals (with respect to arclength)

Ex 2: Evaluate $\int_C 2x\, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

2. Line Integrals (with respect to x and y)

Situation:

- Let C be a curve in the xy-plane with parametrization

$$x = x(t) \ \ , \ \ y = y(t) \ \ , \ \ a \leq t \leq b$$

- Let $f(x, y)$ be a 2-variable function defined on the curve C

2. Line Integrals (with respect to *x* and *y*)

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2. They are called the **line integrals of f along C with respect to x and y**:

5
$$\int_C f(x, y)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

6
$$\int_C f(x, y)\, dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

7
$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$
$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

2. Line Integrals (with respect to *x* and *y*)

Ex 4: Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ is the line segment from (-5,-3) to (0,2) and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from (-5,-3) to (0,2).

2. Line Integrals (with respect to *x* and *y*)

$$\int_{-C} f(x, y)\, dx = -\int_{C} f(x, y)\, dx$$

$$\int_{-C} f(x, y)\, dy = -\int_{C} f(x, y)\, dy$$

$$\int_{-C} f(x, y)\, ds = \int_{C} f(x, y)\, ds$$

3. Line Integrals in Space

9 $$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

3. Line Integrals in Space

<u>Ex 5</u>: Evaluate $\int_C y \sin z \; ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$.

3. Line Integrals in Space

Ex 6: Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from (2,0,0) to (3,4,5), followed by the vertical line segment C_2 from (3,4,5) to (3,4,0).

4. Line Integrals of Vector Fields

Situation:

- Let C be a curve in the xy-plane with parametrization

$$x = x(t) \;\;,\;\; y = y(t) \;\;,\;\; a \leq t \leq b$$

- Let $\vec{F}(x, y)$ be a vector field on $\mathbb{R}^2$ defined on the curve C

4. Line Integrals of Vector Fields

13 Definition

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along** C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Note Even though $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

because the unit tangent vector $\mathbf{T}$ is replaced by its negative when C is replaced by $-C$.

4. Line Integrals of Vector Fields

Ex 7: Find the work done by the force field $\vec{F}(x,y) = < x^2, -xy >$ in moving a particle along the quarter-circle $\vec{r}(t) = < \cos t\ , \sin t >$, $0 \leq t \leq \frac{\pi}{2}$.

4. Line Integrals of Vector Fields

Ex 8: Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) =< xy,\ yz,\ xz >$ and C is the twisted cubic given by $x = t$, $y = t^2$, $z = t^3$, $0 \leq t \leq 1$.